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Separation of variables for scalar evolution equations in one space dimension

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Abstract. The flow of an autonomous kth-order evolution equation in one space dimension is generated by a kth-order ordinary differential operator on the space of fields. The characteristic fields are the solutions of the characteristic equation of the operator. A solution of the evolution equation is separable if and only if it is a curve in the $(k+1)$-parameter space of characteristic fields. The non-stationary characteristic fields with separable evolution are those which remain characteristic under small dilation. Every characteristic field has local separable evolution if and only if the characteristic equation is infinitesimally dilation invariant. This is the case when the evolutionary generator is an infinitesimal symmetry of the characteristic equation, so that its flow stabilizes the space of characteristic fields. The evolution equation is then locally homogeneous if and only if the characteristic value is an invariant of the restricted flow, which is a non-generic property. The separable kth-order evolution equations are parametrized by the kth-order invariants of the locally homogeneous $(k+1)$th-order ordinary differential equations.

1. Introduction

An autonomous scalar evolution equation in one space dimension is a partial differential equation

$$\frac{\partial u}{\partial t} = F\left(x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right) \tag{1}$$

for a single unknown function u of two variables x and t. We assume that F is smooth (C^∞). A solution of equation (1) is a smooth function $u(x,t)$ defined on a connected open subset of the xt-plane with

$$\frac{\partial u}{\partial t}(x,t) = F\left(x, u(x,t), \frac{\partial u}{\partial x}(x,t), \ldots, \frac{\partial^k u}{\partial x^k}(x,t)\right)$$

at all points of its domain. In physical terms, the value $u(x,t)$ represents a measurable quantity at the point x in space at time t, and the equation (1) governs the temporal evolution of the field $x \longmapsto u(x,t)$. This interpretation leads us to require that the domain of a solution be a product of intervals in x and t. Because equation (1) is autonomous, we need only consider solutions defined for t in an interval containing $t = 0$. The relevant initial value problem consists of finding a solution $u(x,t)$ such that

$$u(x,0) = v(x) \tag{2}$$

for given data v. In this study, we are concerned with local properties of solutions, and do not address questions related to boundary conditions or global existence.

A solution $u(x,t)$ is *separable* if it has the form

$$u(x,t) = v(x)w(t) \tag{3}$$

for some functions v and w. The search for separable solutions of partial differential equations has been most effective for linear equations and for special types of first-order nonlinear equations. For example, a summary of the results linking the separable coordinate systems for the Helmholz and Hamilton–Jacobi equations with their symmetry structure is given in Miller [1]. Kalnins and Miller [2] derived conditions for completely general equations which are both necessary and sufficient for the existence of a complete set of separable solutions in a given coordinate system. They used their conditions to characterize regularly separable linear equations [2], and then extended the construction to nonlinear equations [3]. The implications of these results have not been fully developed, however, perhaps because of their complexity and generality. In particular, the geometric significance of the separation conditions is not evident. Various researchers have recently studied separation of variables for special types of higher-order nonlinear equations. For example, Grundland and Infeld [4], Miller and Rubel [5], and Zhdanov [6] studied separability for the one-dimensional (1D) f-Gordon equation, and Doyle and Vassiliou [7] classified the separable 1D nonlinear diffusion equations. In this work, we characterize the separable solutions of the general 1D evolution equation and describe the separation mechanism. We give a geometric interpretation of regular separability, parametrize the class of regularly separable equations, and characterize the subclass of homogeneous equations.

2. Preliminaries

The notation

$$u_t = \frac{\partial u}{\partial t} \qquad u_j = \frac{\partial^j u}{\partial x^j}$$

abbreviates the equation (1) to

$$u_t = F(x, u, u_1, \ldots, u_k). \tag{4}$$

We assume $k \geqslant 1$ and $F_{u_k} \neq 0$, so that (4) is a kth-order differential equation. The kth-order ordinary differential operator

$$F[u] = F(x, u, u_1, \ldots, u_k)$$

is a map on the space of smooth fields which takes $u(x)$ to

$$F[u(x)] = F(x, u(x), u_1(x), \ldots, u_k(x)).$$

Viewing $F[u(x)]\partial_u$ as a vector on the field $u(x)$, a solution $u(x,t)$ of the equation

$$u_t = F[u]. \tag{5}$$

is a smooth curve in the space of fields, tangent at each value of the parameter t to the vector field

$$V = F[u]\partial_u$$

i.e. a solution is a trajectory of V. The initial condition (2) is the requirement that the trajectory pass through the field v at $t = 0$. In general, we cannot claim either that there exists such a trajectory or that it is unique. The singular points of V are the time-independent or *stationary* solutions of (5). They are the solutions $u(x)$ of the kth-order ordinary differential equation

$$F[u] = 0.$$

There is a k-parameter family of stationary solutions, assuming that the solution set of the functional equation

$$F(x, u, u_1, \ldots, u_k) = 0$$

in the space of variables $x, u, u_1, \ldots, u_k$ is non-empty. The autonomous evolutionary form (5) is preserved by arbitrary smooth transformation in x and u, and by affine transformation in t. In the field variable $\bar{u}$, where $\bar{u}(u)$ is a smooth regular function, equation (5) is

$$\bar{u}_t = \bar{F}[\bar{u}]$$

where

$$\bar{F}[\bar{u}] = \bar{u}'(u) F[u]. \tag{6}$$

The relationship (6) is the usual transformation law for the coordinate representation of a vector field, so that

$$F[u]\partial_u = \bar{F}[\bar{u}]\partial_{\bar{u}}.$$

Note that the stationary solutions have intrinsic significance.

3. Homogeneous evolution equations; example of a non-homogeneous separable equation

A linear equation

$$u_t = Lu \tag{7}$$

where

$$L = f_k(x)\partial_x{}^k + \cdots + f_1(x)\partial_x + f_0(x)$$

with $f_k \neq 0$, is separable in the sense that it has many separable solutions. A non-zero function (3) is a solution of (7) if and only if v and w satisfy the ordinary differential equations

$$Lv/v = \lambda \tag{8}$$

and

$$w'/w = \lambda$$

for some constant λ. The stationary solutions correspond to $\lambda = 0$. Equation (7) has a k-dimensional vector space of stationary solutions and a $(k+1)$-parameter family of non-zero separable solutions. Henceforth, we tacitly assume wherever necessary that functions are non-zero.

More generally, equation (5) is *homogeneous* if it has the form

$$u_t = ug(x, u_1/u, \ldots, u_k/u) \tag{9}$$

for some function $g(x, z_1, \ldots, z_k)$ with $g_{z_k} \neq 0$. Homogeneous equations are also separable. A function (3) is a solution of (9) if and only if v and w satisfy the ordinary differential equations

$$g(x, v_1/v, \ldots, v_k/v) = \lambda \tag{10}$$

and

$$w'/w = \lambda$$

for some constant λ. The stationary solutions correspond to $\lambda = 0$. Equation (9) has a k-parameter family of stationary solutions and a $(k+1)$-parameter family of separable solutions.

We now discuss an example where the separation mechanism is more subtle. Fix a regular function $g(z)$ with $g(0) = 0$. A function (3) is a solution of the second-order equation

$$u_t = ug(u - xu_x + x^2u_{xx}/2) \tag{11}$$

if and only if

$$w'(t)/w(t) = g((v(x) - xv'(x) + x^2v''(x)/2)w(t)). \tag{12}$$

Differentiating with respect to x implies the necessary condition

$$v_{xxx} = 0. \tag{13}$$

Conversely, if

$$v(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2$$

then

$$v(x) - xv'(x) + x^2v''(x)/2 = \lambda_0$$

and (12) is a single ordinary differential equation

$$w'/w = g(\lambda_0 w).$$

We find that equation (11) has a three-parameter family of non-stationary separable solutions

$$u(x,t) = (\lambda_0 + \lambda_1 x + \lambda_2 x^2)w(t)$$

where $\lambda_0 \neq 0$ and

$$t = \int_1^w \frac{\mathrm{d}z}{zg(\lambda_0 z)}$$

and a two-parameter family of stationary solutions

$$u(x,t) = \lambda_1 x + \lambda_2 x^2.$$

Note that conditions (8) or (10) or (13) are satisfied if and only if

$$F[v] = \lambda v$$

for some constant λ, where F is the appropriate operator.

4. The characteristic equation; regular separability

A non-zero smooth function $u(x,t)$, defined on a product of intervals x and t has form (3) if and only if it satisfies the partial differential equation

$$u_{xt} = u_x u_t/u. \tag{14}$$

Hence the separable solutions of (5) are the solutions of the overdetermined system of equations (5) and (14). The stationary solutions of (5) satisfy (14), for any operator F, but we do not generally expect joint solutions which depend on t. If $u(x,t)$ is a solution of (5) then

$$\frac{\partial}{\partial x}\left(\frac{F[u(x,t)]}{u(x,t)}\right) = \frac{1}{u(x,t)}\left(\frac{\partial^2 u(x,t)}{\partial x \partial t} - \frac{1}{u(x,t)}\frac{\partial u(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial t}\right)$$

so the solution is separable if and only if it satisfies the differential equation

$$D_x(F[u]/u) = 0 \tag{15}$$

where

$$D_x = \partial_x + u_1\partial_u + u_2\partial_{u_1} + \cdots$$

is the total x-derivative. Note that the variable t enters into (15) as a parameter. Disregarding t, we view the *characteristic equation* (15) as a $(k+1)$th-order ordinary differential equation. A solution of (5) is separable if and only if it is a one-parameter family of solutions of the characteristic equation of the operator F.

Theorem 1. A non-zero solution $u(x, t)$ of the equation

$$u_t = F[u] \tag{16}$$

is separable if and only if the function $x \longmapsto u(x, t)$ is a solution of the ordinary differential equation

$$D_x(F[u]/u) = 0$$

for each value of t.

A *characteristic field* of the operator F is a non-zero function $u(x)$ such that

$$F[u(x)] = \lambda u(x)$$

for some constant λ, the *characteristic value* of the field. For example, the characteristic fields with value $\lambda = 0$ are the non-zero stationary solutions of (5). The characteristic fields of F are the solutions of the characteristic equation (15). Theorem 1 shows that a solution of (16) is separable if and only if it is a curve in the $(k+1)$-parameter space of characteristic fields of F. In particular, the initial value of any separable solution is characteristic. Which characteristic fields are initial values of separable solutions? A characteristic field $u(x)$ is *homogeneous* if αu is characteristic for every value of α in some interval containing $\alpha = 1$, i.e. if small dilations of u are also characteristic.

Theorem 2. Fix a characteristic field v of the operator F, with non-zero characteristic value. There is a separable solution $u(x, t)$ of the initial value problem

$$u_t = F[u] \qquad u(x, 0) = v(x) \tag{17}$$

if and only if v is homogeneous, in which case there is an $\varepsilon > 0$ such that $u(x, t) = v(x)w(t)$ for $|t| < \varepsilon$, where w is the unique solution of the initial value problem

$$w' = W(w) \qquad w(0) = 1 \tag{18}$$

where

$$W(w) = F[v(x)w]/v(x).$$

Proof. If v is homogeneous then

$$F[v(x)w]/v(x)$$

depends only on the parameter w, for values near $w = 1$, and $u(x, t) = v(x)w(t)$ satisfies (17) if w satisfies (18). Conversely, if $u(x, t)$ is a separable solution of (17) then $u(x, t) = v(x)w(t)$, where

$$w'(t) = F[v(x)w(t)]/v(x) \qquad w'(0) = 1.$$

Each non-zero field $vw(t)$ is characteristic, by theorem 1. Note that $w'(0) \neq 0$, because the characteristic value of v is nonzero. Therefore v is homogeneous, and $w(t)$ satisfies (18), for small values of t. □

For example, the characteristic equation of the operator

$$F[u] = u_{xx} - 2uu_x^2$$

is

$$u_{xxx} = u_x u_{xx}/u + 4uu_x u_{xx}$$

and the homogeneous characteristic fields of F are the linear functions. The characteristic value of the field $v(x) = x$ is $\lambda = -2$. The solution of the initial value problem

$$w' = F[xw]/x \qquad w(0) = 1$$

is

$$w(t) = 1/\sqrt{4t+1}$$

so the unique separable solution of the equation

$$u_t = u_{xx} - 2uu_x^2 \tag{19}$$

with $u(x, 0) = x$ is

$$u(x, t) = x/\sqrt{4t+1}. \tag{20}$$

The change of variable $u \longmapsto \mathrm{erf}(u)$ transforms (19) to the linear diffusion equation

$$u_t = u_{xx} \tag{21}$$

and transforms (20) to a t-translate of the similarity solution

$$u(x, t) = \mathrm{erf}(x/2\sqrt{t}).$$

All solutions

$$u(x) = \pm \mathrm{e}^{\lambda_0 + \lambda_1 x + \lambda_2 x^2}$$

of the characteristic equation

$$u_{xxx} = 3u_x u_{xx}/u - 2u_x^3/u^2$$

of the operator

$$F[u] = u_{xx} - u_x^2/u$$

are homogeneous. The characteristic value of the field $v(x) = \mathrm{e}^{x^2}$ is $\lambda = 2$. The solution of the initial value problem

$$w' = F[\mathrm{e}^{x^2} w]/\mathrm{e}^{x^2} \qquad w(0) = 1$$

is

$$w(t) = \mathrm{e}^{2t}$$

so the unique separable solution of the equation

$$u_t = u_{xx} - u_x^2/u \tag{22}$$

with $u(x, 0) = \mathrm{e}^{x^2}$ is

$$u(x, t) = \mathrm{e}^{x^2 + 2t}. \tag{23}$$

The change of variable $u \longmapsto \ln u$ transforms (22) to the linear equation (21), and transforms (23) to the heat polynomial

$$u(x, t) = x^2 + 2t.$$

Equation (16) is *regularly separable* [2] if there is a separable solution of the initial value problem (17) on a neighbourhood of x, for each x in the domain of v, for each characteristic field v. This holds if and only if every non-stationary characteristic field of F is locally homogeneous, by theorem 2. Hence (16) is regularly separable when dilation of the dependent variable u is a local symmetry of (15), i.e. when its charastic equation is infinitesimally dilation invariant. The dilation invariant $(k+1)$th-order ordinary differential equations are those which locally have the *homogeneous* form

$$u_{k+1} = u\phi(x, u_1/u, \ldots, u_k/u).$$

See Bluman and Kumei [8] or Olver [9, 10] for a discussion of symmetry and differential equations. Theorem 3 is equivalent to theorem 4, which we prove carefully.

Theorem 3. The kth-order equation

$$u_t = F[u]$$

is regularly separable if and only if its characteristic equation

$$D_x(F[u]/u) = 0$$

is locally homogeneous.

A homogeneous equation (9) is regularly separable, because its characteristic equation

$$D_x(g(x, u_1/u, \ldots, u_k/u)) = 0$$

is homogeneous. Equation (11) is not homogeneous, but is nevertheless regularly separable, because its characteristic equation (13) is homogeneous. Equation (22) is homogeneous, hence regularly separable. Equation (19) is not regularly separable.

5. Geometric formulation

We now cast the problem into differential geometric form. The construction is invariant, but easiest to describe in the field variable which simplifies the separability condition. For either $u > 0$ or $u < 0$, the change of variable $u \longmapsto \ln|u|$ transforms (14) to

$$u_{xt} = 0. \tag{24}$$

The solutions of (24) are the functions with *additively separable* form

$$u(x, t) = v(x) + w(t).$$

The additively separable solutions of the equation

$$u_t = F[u] \tag{25}$$

are the joint solutions of (24) and (25). The natural geometric context for studying the compatibility of these equations is a structure which incorporates the independent and dependent variables and the derivatives of the dependent variables. The variables x, t, and $u_{i,j}$, with $i, j \geqslant 0$ and $i + j \leqslant k$, are coordinates on the bundle J^k of kth-order jets of smooth mappings $\mathbb{R}^2 \longrightarrow \mathbb{R}$, where $u_{i,j}$ represents the $(i+j)$th derivative $\partial^{i+j}u/\partial x^i \partial t^j$ of the variable $u = u_{0,0}$. For example, $u_{i,0} = u_i$, $u_{0,1} = u_t$, and $u_{1,1} = u_{xt}$. See Olver [10] for the basic facts about jet bundles, contact structure, and prolongation, and Olver [11] for discussion of differential constraints and compatibility conditions.

We begin with the anomalous case of the first-order equation

$$u_t = F(x, u, u_x). \tag{26}$$

Differentiating (26), we find that the joint solutions of (24) and (26) are the solutions of the system

$$\begin{aligned} u_t &= F(x,u,u_x) \\ u_{tt} &= FF_u(x,u,u_x) \\ u_{xt} &= 0 \\ u_{xx} &= -(F_x + u_x F_u)/F_{u_x}(x,u,u_x). \end{aligned} \tag{27}$$

The eight-dimensional manifold J^2 is parametrized by the variables $x, t, u, u_x, u_t, u_{xx}, u_{xt}$, and u_{tt}, and relations (27) define a four-dimensional submanifold $\mathcal{R}$, parametrized by x, t, u, and u_x. The pullback C of the contact system on J^2 to $\mathcal{R}$ is spanned by the independent 1-forms

$$\theta = \mathrm{d}u - u_x\,\mathrm{d}x - F\,\mathrm{d}t$$

and

$$\theta_x = \mathrm{d}u_x + (F_x + u_x F_u)/F_{u_x}\,\mathrm{d}x.$$

The two-dimensional integrals of C are (locally) the second-order prolongations of the solutions of (27). Note that C is two-codimensional, regardless of F, and hence has at most a two-parameter family of two-dimensional integrals, so that (26) has at most a two-parameter family of additively separable solutions. The maximal case occurs when C is integrable. We have

$$\mathrm{d}\theta \equiv 0 \mod C$$

and

$$\mathrm{d}\theta_x \equiv F((F_x + u_x F_u)/F_{u_x})_u\,\mathrm{d}t \wedge \mathrm{d}x \mod C$$

so C is integrable if and only if

$$((F_x + u_x F_u)/F_{u_x})_u = 0. \tag{28}$$

Proceeding to the cases $k \geqslant 2$, we find by differentiating (25) with respect to t that the joint solutions of (24) and (25) are the solutions of the system

$$\begin{aligned} u_{0,j} &= (F\partial_u)^{j-1}F & j &= 1,\ldots,k, \\ u_{i,j} &= 0 & i, j &\geqslant 1, \quad i+j \leqslant k. \end{aligned} \tag{29}$$

The relations (29) define a $(k+3)$-dimensional submanifold $\mathcal{R}$ of J^k, parametrized by the variables $x, t, u, u_1, \ldots, u_k$. The pullback C of the contact system on J^k to $\mathcal{R}$ is spanned by the 1-forms

$$\begin{aligned} \theta_0 &= \mathrm{d}u - u_1\,\mathrm{d}x - u_{0,1}\,\mathrm{d}t \\ \theta_i &= \mathrm{d}u_i - u_{i+1}\,\mathrm{d}x & i &= 1,\ldots,k-1 \\ \phi_j &= \mathrm{d}u_{0,j} - u_{0,j+1}\,\mathrm{d}t & j &= 1,\ldots,k-1 \end{aligned}$$

with the t-derivatives of u given by (29). The 2-dimensional integrals of C are the kth-order prolongations of the solutions of (29). We have

$$\phi_1 \equiv F_{u_k}\theta_k \mod \theta_0,\ldots,\theta_{k-1}$$

where

$$\theta_k = \mathrm{d}u_k + \tilde{D}_x F/F_{u_k}\,\mathrm{d}x$$

with

$$\tilde{D}_x = \partial_x + u_1\partial_u + u_2\partial_{u_1} + \cdots + u_k\partial_{u_{k-1}}$$

so C is spanned by $\theta_0, \ldots, \theta_k, \phi_2, \ldots, \phi_{k-1}$. Note that $\theta_0, \ldots, \theta_k$ are independent. This implies that C is at most two-codimensional, and hence has at most a $(k+1)$-parameter family of two-dimensional integrals, so that (25) has at most a $(k+1)$- parameter family of additively separable solutions. Equations (24) and (25) are *compatible* in the maximal case when C is both two-codimensional and integrable.

Theorem 4. The kth-order equation

$$u_t = F[u] \tag{30}$$

and the additive separability condition

$$u_{xt} = 0 \tag{31}$$

are compatible if and only if

$$(\tilde{D}_x F/F_{u_k})_u = 0. \tag{32}$$

Proof. Condition (32) in the case $k = 1$ is (28). For $k = 2$, the system (29) is

$$u_t = F$$
$$u_{tt} = FF_u$$
$$u_{xt} = 0.$$

The five-dimensional manifold $\mathcal{R}$ is parametrized by the variables x, t, u, u_x, and u_{xx} and C is spanned by the independent 1-forms

$$\theta = \mathrm{d}u - u_x\mathrm{d}x - F\mathrm{d}t$$
$$\theta_x = \mathrm{d}u_x - u_{xx}\mathrm{d}x$$
$$\theta_{xx} = \mathrm{d}u_{xx} + \tilde{D}_x F/F_{u_{xx}}\mathrm{d}x$$

where

$$\tilde{D}_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x}.$$

In this case, C is two-codimensional, regardless of F. We have

$$\mathrm{d}\theta, \mathrm{d}\theta_x \equiv 0 \mod C$$

and

$$\mathrm{d}\theta_{xx} \equiv F(\tilde{D}_x F/F_{u_{xx}})_u \mathrm{d}t \wedge \mathrm{d}x \mod C$$

so C is integrable if and only if (32) holds. Now suppose $k \geqslant 3$. We will show that C is two-codimensional if and only if (32) holds, in which case C is integrable. Note that C is two-codimensional if and only if

$$\phi_2, \ldots, \phi_{k-1} \equiv 0 \mod \theta.$$

We have

$$\phi_j \equiv (\tilde{D}_x u_{0,j} - (u_{0,j})_{u_k}\tilde{D}_x F/F_{u_k})\mathrm{d}x \mod \theta$$

so

$$\phi_j \equiv 0 \mod \theta \tag{33}$$

if and only if

$$\tilde{D}_x u_{0,j} - (u_{0,j})_{u_k} \tilde{D}_x F / F_{u_k} = 0. \tag{34}$$

For example,

$$\phi_2 \equiv 0 \quad \mathrm{mod}\ \theta$$

if and only if

$$\tilde{D}_x(F F_u) - (F F_u)_{u_k} \tilde{D}_x F / F_{u_k} = 0$$

i.e.

$$F(\tilde{D}_x F_u - F_{u u_k} \tilde{D}_x F / F_{u_k}) = 0.$$

The latter condition is equivalent to (32). Hence (32) holds if C is two-codimensional. Conversely, suppose that (32) holds. If (33) holds for some $j \geqslant 2$, as just verified for $j = 2$, then using (34) we find

$$\begin{aligned}
\tilde{D}_x u_{0,j+1} &- (u_{0,j+1})_{u_k} \tilde{D}_x F / F_{u_k} \\
&= \tilde{D}_x(F(u_{0,j})_u) - (F(u_{0,j})_u)_{u_k} \tilde{D}_x F / F_{u_k} \\
&= F(\tilde{D}_x (u_{0,j})_u - (u_{0,j})_{u u_k} \tilde{D}_x F / F_{u_k}) \\
&= F(((u_{0,j})_{u_k} \tilde{D}_x F / F_{u_k})_u - (u_{0,j})_{u u_k} \tilde{D}_x F / F_{u_k}) \\
&= F(u_{0,j})_{u_k} (\tilde{D}_x F / F_{u_k})_u = 0
\end{aligned}$$

so that

$$\phi_{j+1} \equiv 0 \quad \mathrm{mod}\ \theta.$$

This proves that C is two-codimensional if and only if (32) holds. In this case, C is also integrable, because

$$\mathrm{d}\theta_0, \ldots, \mathrm{d}\theta_{k-1} \equiv 0 \quad \mathrm{mod}\ \theta \tag{35}$$

and

$$\mathrm{d}\theta_k \equiv F(\tilde{D}_x F / F_{u_k})_u \, \mathrm{d}t \wedge \mathrm{d}x \quad \mathrm{mod}\ \theta. \qquad \square$$

The zero set of F is a $(k+2)$-dimensional submanifold of $\mathcal{R}$ (assuming that it is non-empty) on which the 1-forms ϕ are equal to zero, so C restricts to a two-codimensional differential system spanned by $\theta_0, \ldots, \theta_{k-1}$, and the restriction is integrable, by (35). Hence the zero set of F is foliated by a k-parameter family of additively separable solutions of (30), whether or not (32) holds. These are the stationary solutions of (30). The compatibility condition (32) is required for the existence of a $(k+1)$-parameter family of additively separable solutions.

The necessary vanishing of θ_k on an additively separable solution $u(x, t)$ shows that the function $x \longmapsto u(x, t)$ satisfies the ordinary differential equation

$$D_x F[u] = 0 \tag{36}$$

for each value of t. This implies that C is two-codimensional and integrable if and only if there is an additively separable solution of the initial value problem

$$u_t = F[u] \qquad u(x, 0) = v(x)$$

on a neighbourhood of x, for each x in the domain of v, for each solution v of (36) (note that C is invariant under translation in t). In the variable $\bar{u} = \mathrm{e}^u$, equation (30) is

$$\bar{u}_t = \bar{F}[\bar{u}] \tag{37}$$

where $F[u] = \bar{F}[\bar{u}]/\bar{u}$, and (36) is the characteristic equation

$$D_x(\bar{F}[\bar{u}]/\bar{u}) = 0 \tag{38}$$

of the operator $\bar{F}$. Hence C is two-codimensional and integrable if and only if (37) is regularly separable. The condition (32) holds if and only if the vector field ∂_u is an infinitesimal symmetry of (36), i.e. if and only if the dilation generator $\bar{u}\partial_{\bar{u}}$ is an infinitesimal symmetry of (38). Therefore theorem 3 and theorem 4 are equivalent. The condition (32) is the regular separability condition of Kalnins and Miller [2] for 1D evolution equations.

For example, for the second-order equation

$$u_t = f(u)u_{xx} + g(u)u_x^2 \tag{39}$$

we have

$$(\tilde{D}_x F/F_{u_{xx}})_u = ((f' + 2g)/f)'u_x u_{xx} + (g'/f)'u_x^3$$

so (32) holds if and only if

$$((f' + 2g)/f)' = 0 \qquad (g'/f)' = 0.$$

This system of ordinary differential equations is easily solved. Transforming the result into the field variable in which (39) has the canonical form

$$u_t = (k(u)u_x)_x \tag{40}$$

we find all functions $k(u)$ such that the diffusion equation (40) is regularly separable in some variable [7]. The separable diffusion equations with translation invariant source terms linear in u_x can be classified in the same way.

The Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x$$

is given in the variable $\bar{u}$ by

$$\bar{u}_t = \bar{u}_{xxx} + 3(u''(\bar{u})/u'(\bar{u}))\bar{u}_x\bar{u}_{xx} + (u'''(\bar{u})/u'(\bar{u}))\bar{u}_x^3 + u(\bar{u})\bar{u}_x.$$

The condition

$$(\tilde{D}_x\bar{F}/\bar{F}_{\bar{u}_{xxx}})_{\bar{u}} = 0$$

is never satisfied, so there is no variable in which the KdV equation is regularly separable. Note, however, that $u(x) = x$ is a homogeneous characteristic field of the operator

$$F[u] = u_{xxx} + uu_x$$

leading to the separable solution

$$u(x, t) = x/(1 - t).$$

Suppose that (30) and (31) are compatible, so that $\mathcal{R}$ is foliated by the two-dimensional integrals of C. The independent commuting vector fields

$$X = \tilde{D}_x - (\tilde{D}_x F/F_{u_k})\partial_{u_k} \qquad T = \partial_t + F\partial_u$$

are annihilated by C, hence are tangent to each integral. The additively separable solutions of (30) are thus identified with the one-parameter families of trajectories of X generated by the flow of T. The trajectories of X are the t-translates of the solutions of (36). In this construction, T generates the flow of the evolution equation (30) on the solution space of the ordinary differential equation (36).

Theorem 5. The equation

$$u_t = F[u]$$

is regularly separable if and only if its generator

$$V = F[u]\partial_u$$

is an infinitesimal symmetry of its characteristic equation

$$D_x(F[u]/u) = 0.$$

Proof. See Olver [9] for discussion of generalized symmetry. In the additive formulation, the characteristic equation is (36). We have

$$V(D_x F) = \sum_{i=0}^{k+1} (D_x{}^i F)(D_x F)_{u_i}$$

so

$$V(D_x F) \equiv 0 \quad \text{mod } D_x F = 0$$

if and only if

$$(D_x F)_u \equiv 0 \quad \text{mod } D_x F = 0$$

i.e. if and only if (32) holds. □

The result is analogous to Svirshchevskii's formulation [12] of the reduction method of Galaktionov [13] and colleagues in terms of generalized symmetries of linear ordinary differential equations.

6. Parametrization of the class of regularly separable equations

We now write $F[u] = uG[u]$, where $G(x, u, u_1, \ldots, u_k)$ is a smooth function with $G_{u_k} \neq 0$. The characteristic equation

$$D_x G[u] = 0 \tag{41}$$

is the unique $(k+1)$th-order ordinary differential equation for which G is an *invariant* (i.e. first integral; note that all first integrals of an nth-order ordinary differential equation have order $n-1$).

Theorem 6. The equation

$$u_t = uG[u] \tag{42}$$

is regularly separable if and only if G is an invariant of a locally homogeneous ordinary differential equation.

The regularly separable kth-order evolution equations are thus in one-to-one correspondence with the kth-order invariants of the locally homogeneous $(k+1)$th-order ordinary differential equations. The solutions of a given locally homogeneous ordinary differential equation are the initial values of the separable solutions of each associated evolution equation.

7. Characterization of homogeneity

The generator of the homogeneous ordinary differential equation

$$u_{k+1} = u\phi(x, u_1/u, \ldots, u_k/u) \tag{43}$$

is the vector field

$$X_\phi = \tilde{D}_x + u\phi(x, u_1/u, \ldots, u_k/u)\partial_{u_k}$$

on the space $(x, u, u_1, \ldots, u_k)$, and its invariants are the non-constant functions $G(x, u, u_1, \ldots, u_k)$ such that

$$X_\phi G = 0. \tag{44}$$

Note that (41) and (43) are identical if and only if G satisfies (44). The local solutions of the partial differential equation (44) are arbitrary functional combinations of a complete set of independent solutions $\zeta_0, \ldots, \zeta_k$. Hence, locally, the evolution equations (42) with characteristic equation (43) are those with $G = g(\zeta_0, \ldots, \zeta_k)$ for some function g. We can assume that $\zeta_0, \ldots, \zeta_{k-1}$ are joint invariants of the independent commuting vector fields X_ϕ and

$$U = u\partial_u + u_1\partial_{u_1} + \cdots + u_k\partial_{u_k}$$

so that they depend only on the variables $x, u_1/u, \ldots, u_k/u$. The general joint invariant is an arbitrary functional combination of $\zeta_0, \ldots, \zeta_{k-1}$. Equation (42) is locally homogeneous if and only if G is an invariant of U. Hence the homogeneous evolution equations (42) with characteristic equation (43) are those with $G = g(\zeta_0, \ldots, \zeta_{k-1})$. This shows that the generic regularly separable equation is non-homogeneous. If $v(x)$ is a solution of (43) then $v(x)w$ is locally a one-parameter family of solutions. The invariants ζ are constant on the solutions of (43), so

$$\zeta_j[v(x)w] = \alpha_j(w)$$

for some functions $\alpha_0, \ldots, \alpha_k$. Note that $\alpha_0, \ldots, \alpha_{k-1}$ do not depend on w, because $\zeta_0, \ldots, \zeta_{k-1}$ are homogeneous. Hence the variation of the characteristic value

$$\lambda(w) = g(\alpha_0, \ldots, \alpha_{k-1}, \alpha_k(w))$$

is entirely due to the presence of the non-homogeneous invariant ζ_k, suggesting the following characterization of homogeneity.

Theorem 7. A regularly separable equation

$$u_t = F[u] \tag{45}$$

is locally homogeneous if and only if the characteristic value of every separable solution is independent of t.

Proof. In the additive formulation, the vector field $T = \partial_t + F\partial_u$ generates the separable evolution of characteristic fields, and their characteristic values are given by F. $\mathcal{R}$ is foliated by the family of separable solutions. Hence the characteristic value of every separable solution is constant if and only if F is invariant under the flow of T, i.e. if and only if $F_u = 0$. The multiplicative version of this condition is local homogeneity. □

The space of characteristic fields of the operator F is foliated by the hyperspaces of constant characteristic value. These are the solution spaces of the kth-order ordinary differential equations

$$F[u]/u = \lambda. \tag{46}$$

The flow of a regularly separable equation restricts to the space of characteristic fields. Theorem 7 shows that the equation is locally homogeneous if and only if the flow preserves the characteristic value foliation. As in theorem 5, it can be shown that (45) is locally homogeneous if and only if its generator is an infinitesimal symmetry of every equation (46).

8. Examples

The solutions of the homogenous third-order equation

$$u_{xxx} = u_x u_{xx}/u \tag{47}$$

are the solutions of the second-order equations

$$u_{xx}/u = \lambda$$

for the various values of λ. Phase, frequency, and amplitude are independent invariants of (47). Phase and frequency are homogeneous, and amplitude is non-homogeneous. Phase depends on x. Frequency and amplitude are combinations of the independent invariants

$$u_{xx}/u \qquad uu_{xx} - {u_x}^2$$

and do not depend on x. A translation invariant second-order evolution equation has characteristic equation (47) if and only if it has the form

$$u_t = ug(u_{xx}/u, uu_{xx} - u_x^2) \tag{48}$$

for some function $g(\zeta_1, \zeta_2)$. These are the only translation invariant second-order equations which propagate arbitrary exponential data by dilation. Equation (48) is homogenous if and only if $g_{\zeta_2} = 0$. For example, we obtain the linear diffusion equation (21) in the case $g(\zeta_1, \zeta_2) = \zeta_1$. If $g(\zeta_1, \zeta_2) = \zeta_1 + \zeta_2$ then (48) is the non-homogeneous regularly separable equation

$$u_t = (1 + u^2)u_{xx} - uu_x^2$$

which is transformed to the nonlinear diffusion equation

$$u_t = (\tanh^{-1} u)_{xx} \tag{49}$$

by the change of variable $u \longmapsto u/\sqrt{1+u^2}$. See Doyle and Vassiliou [7] for the explicit separable solutions of (49) thus obtained.

We conclude with a description of the kth-order equations which propagate arbitrary kth-order degree polynomial data by dilation. The independent functions

$$\zeta_j = \sum_{i=j}^{k} \frac{(-1)^{i+j}}{j!(i-j)!} x^{i-j} u_i \qquad j = 0, \ldots, k$$

are invariants of the linear equation

$$u_{k+1} = 0. \tag{50}$$

In fact,

$$\zeta_j[\lambda_0 + \lambda_1 x + \cdots + \lambda_k x^k] = \lambda_j.$$

See Svirshchevskii [12] for a complete description of the invarants and symmetries of linear ordinary differential equations. An evolution equation has characteristic equation (50) if and only if it has the form

$$u_t = ug(\zeta_0, \ldots, \zeta_k). \tag{51}$$

The separable solutions of (51) are the functions

$$u(x,t) = (\lambda_0 + \lambda_1 x + \cdots + \lambda_k x^k) w(t) \tag{52}$$

where

$$w'/w = g(\lambda_0 w, \ldots, \lambda_k w).$$

The characteristic value of (52) is

$$\lambda(t) = g\big(\lambda_0 w(t), \ldots, \lambda_k w(t)).$$

The characteristic value of every separable solution of (51) is constant if and only if $g\big(\lambda_0 w, \ldots, \lambda_k w\big)$ is independent of local variation in w, for each set of values $\lambda_0, \ldots, \lambda_k$. This holds if and only if g is locally a homogeneous function of the linear invariants $\zeta_0, \ldots, \zeta_k$, i.e. if and only if (51) is locally homogeneous. Equation (11) is a non-homogeneous example in the case $k = 2$.

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